B.Sc (Math) part-I

paper — II

Topic: Notion of Integral as limit of sum

we define $\int_a^b f(x)\,dx = \lim_{h \to 0} \sum_{r=1}^{n} f(a+rh)$

or $\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + \cdots + f(a+\overline{n-1}\,h)]$

where $nh = b-a$

problem Evaluate from first principle

$\int_a^b e^{mx}\,dx$

Soln:— From the def

$\int_a^b f(x)\,dx = \underset{h \to 0}{Lt}\, h[f(a) + f(a+h) + \cdots + f(a+\overline{n-1})h]$

where $nh = b-a$

Here $f(x) = e^{mx}$ Therefore

$\int_0^b e^{mx}\,dx = \underset{h \to 0}{Lt}\, h[e^{ma} + e^{m(a+h)} + e^{m(a+2h)} + \cdots \text{ to } n \text{ term}]$

$= \underset{h \to 0}{Lt}\, h[e^{ma} + e^{ma}e^{mh} + e^{ma}e^{2mh} + \cdots \text{ to } n \text{ term}]$

$= \underset{h \to 0}{Lt}\, h e^{ma}[1 + e^{mh} + e^{2mh} + \cdots \text{ to } n \text{ term}]$

$= \underset{h \to 0}{Lt}\, h e^{ma}\left[\frac{1((e^{mh})^n - 1)}{e^{mh} - 1}\right]$

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$= \underset{h\to 0}{Lt}\, h e^{ma}\left[\frac{(e^{m})^{nh}-1}{e^{mh}-1}\right]$

$= \underset{h\to 0}{Lt}\, [e^{m(b-a)}-1] \times \frac{h}{e^{mh}-1}$

$= e^{ma}[e^{m(b-a)}-1] \times \underset{h\to 0}{Lt}\, \frac{h}{e^{mh}-1}$

$= (e^{mb}-e^{ma}) \times \underset{h\to 0}{Lt}\, \frac{1}{me^{mh}} = (e^{mb}-e^{ma}) \times \frac{1}{m}$

$= \frac{e^{mb}-e^{ma}}{m}$

problem (2) $\int_0^{\pi/2} \cos x \, dx$

Soln:- From the def

$\int_a^b f(x)\,dx = \underset{h\to 0}{Lt}\, h[f(a+h) + f(a+2h) + \cdots + \text{to } n \text{ term}]$

let $a=0$, $b=\pi/2$ so that

$\int_0^{\pi/2} f(x)\,dx = \underset{h\to 0}{Lt}\, h[f(h) + f(2h) + \cdots \text{ to } n \text{ term}]$

where $nh = b-a = \pi/2$

suppose that $f(x) = \cos x$ Therefore

$\int_0^{\pi/2} \cos x\,dx = \underset{h\to 0}{Lt}\, h[\cos h + \cos 2h + \cdots \text{ to } n \text{ term}]$

$= \underset{h\to 0}{Lt}\, h\left[\frac{\sin \frac{nh}{2}}{\sin \frac{h}{2}} \cos\left(h + \frac{n-1}{2}h\right)\right]$

$= \underset{h\to 0}{Lt}\, h\left[\frac{\sin \frac{\pi}{4}}{\sin \frac{h}{2}} \cos\left(\frac{nh+h}{2}\right)\right]$

$= \sin\frac{\pi}{4} \times \underset{h\to 0}{Lt} \frac{\frac{h}{2}\times 2}{\sin\frac{h}{2}} \times \cos\frac{\frac{\pi}{2}+h}{2}$

$= \sin\frac{\pi}{4} \times 2 \times \underset{h\to 0}{Lt} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \times \underset{h\to 0}{Lt} \cos\frac{\frac{\pi}{2}+h}{2}$

$= \sin\frac{\pi}{4} \times 2 \times 1 \times \cos\frac{\pi}{4}$

$= 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} \times 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$

$= 1$

(3) Evaluate $\int_a^b x^2\,dx$ by summation

Soln:- Let $f(x) = x^2$

$\therefore f(a) = a^2,\ f(a+h) = (a+h)^2,$
$f(a+2h) = (a+2h)^2, \ldots$
$f(a+\overline{n-1}\,h) = (a+(\overline{n-1})h)^2$

From $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} h f(a+rh)$ we have

$\int_a^b x^2\,dx = \lim_{n\to\infty} h\left[a^2 + (a+h)^2 + (a+2h)^2 + \ldots + (a+(\overline{n-1})h)^2\right]$

where $nh = b-a$ i.e. $h = \frac{b-a}{n}$

from which $n\to\infty$ therefore $h\to 0$

$= \underset{h\to 0}{Lt}\ h[na^2 + 2ah(1+2+3+\ldots+n$

$$+h^2\left(1^2+2^2+\cdots+\overline{n-1}^2\right)\Big]\Big]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah\,\frac{n(n-1)}{2} + h^2\,\frac{1}{6}\,n(n-1)(2n-1)\right]$$

$$= \lim_{h\to 0}\left[nha^2 + nh(n-1)h\cdot a + \frac{1}{3}\,nh(nh-h)\left(nh-\frac{h}{2}\right)\right]$$

$$= \left[(b-a)\,a^2 + (b-a)(b-a-0)\,a + \frac{1}{3}\,nh(nh-h)\left(nh-\frac{h}{2}\right)\right]$$

$$= \left[(b-a)a^2 + (b-a)(b-a-0)a + \frac{1}{2}(b-a-0)\cdot(b-a-0)\right]$$

$$= \frac{1}{2}(b-a)\left(3a^2+3ab-3a^2+b^2-2ab+a^2\right)$$

$$= \frac{1}{2}(b-a)\left(b^2+ab+a^2\right)$$

$$= \frac{1}{2}\left(b^2-a^2\right)$$